# $2B_p$ AND $4B_p$ ARE TOPOLOGICALLY CONJUGATE

BINGZHE HOU, GONGFU LIAO, AND YANG CAO

Abstract. Let $\lambda B_p$, where $\lambda$ is a nonzero complex number, denote a constant-weighted backward shift operators on $l^p$ for $1 \leq p < \infty$. In this article, we investigate, in topologically conjugacy, the complete classification for $\lambda B_p$.

## 1. Introduction and preliminaries

A discrete dynamical system is simply a continuous function $f : X \to X$ where $X$ is a complete separable metric space. For $x \in X$, the orbit of $x$ under $f$ is $Orb(f, x) = \{x, f(x), f^2(x), \dots\}$ where $f^n = f \circ f \circ \cdots \circ f$ is the $n^{th}$ iterate of $f$ obtained by composing $f$ with $n$ times. A fundamental but difficult problem is the classification for dynamical systems in the sense of "topological conjugacy". If $f : X \to X$ and $g : Y \to Y$ are two continuous mappings, then $f$ is topologically conjugate to $g$ if there exists a homeomorphism $h : X \to Y$ such that $g = h \circ f \circ h^{-1}$, we also say that $h$ conjugates $f$ to $g$. Notice that "topological conjugacy" is an equivalent relation. Moreover, it is easy to see that such many properties as, periodic density, transitivity, mixing $\dots$, are preserved under topological conjugacy.

Recall that $f$ is transitive if for any two non-empty open sets $U, V$ in $X$, there exists an integer $n \geq 1$ such that $f^n(U) \cap V \neq \phi$. It is well known that, in a complete metric space without isolated points, transitivity is equivalent to the existence of dense orbit ([12]). $f$ is strongly mixing if for any two non-empty open sets $U, V$ in $X$, there exists an integer $m \geq 1$ such that $f^n(U) \cap V \neq \phi$ for every $n \geq m$. $f$ has sensitive dependence on initial conditions (or simply $f$ is sensitive)if there is a constant $\delta > 0$ such that for any $x \in X$ and any neighborhood $U$ of $x$, there exists a point $y \in X$ such that $d(f^n(x), f^n(y)) > \delta$, where $d$ denotes the metric on $X$. Moreover, following Devaney [3], $f$ is chaotic if $(a)$ the periodic points for $f$ are dense in $X$, $(b)$ $f$ is transitive, and $(c)$ $f$ has sensitive dependence on initial conditions. It was shown by Banks et. al. ([1]) that $(a) + (b)$ implies $(c)$ and hence chaoticity is preserved under topological conjugacy, though sensitivity is not. For more relative results, we refer to [2] and [3].

We are interested in the dynamical systems induced by continuous linear operators on Banach spaces. In recent years, there has been got some improvements at this aspect (Grosse-Erdmann's and Shapiro's articles [5, 10] are good surveys.).

*Date*: October 22, 2008.

2000 *Mathematics Subject Classification.* Primary 37C15; Secondary 47B37.

*Key words and phrases.* topological conjugacy, weighted backward shift operators, homeomorphism.

This work was supported by the National Nature Science Foundation of China (Grant No. 10771084). The first author is supported by the Youth Foundation of Institute of Mathematics, Jilin university.

In operator theory, we have the concept "similarity"; two operators $A$ and $T$, on Banach spaces $\mathcal{H}$ and $\mathcal{K}$, are similar if there is a bounded linear transformation $S$ from $\mathcal{H}$ onto $\mathcal{K}$ with a bounded inverse, such that $A = S^{-1}TS$. Obviously, it is a stronger relation than topological conjugacy, i.e., if two operators are similar then they must be topologically conjugate. Our ultimate aim is to give a complete classification for continuous linear operators in the topologically conjugate sense. In the present paper, we restrict our attention to weighted backward shift operators on $l^p$ for $1 \leq p < \infty$. Here $l^p$ is the classical Banach space of absolutely $p^{th}$ power summable sequences $x = (x_1, x_2, \dots)$ and we use $\| \cdot \|_p$ to represent its norm. Without confusion, we also use 0 to denote the zero point of $l^p$. Let $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ be a bounded sequence of nonzero complex numbers and let $\pi_n$ be the projection from $l^p$ to the $n^{th}$ coordinate, i.e., $\pi_n(x) = x_n$ for $n \geq 1$. The weighted backward shift operator $\Lambda B_p$ is defined on $l^p$ by

$$\pi_n \circ \Lambda B_p(x) = \lambda_n x_{n+1}, \quad for\ all\ n \geq 1,$$

where $x = (x_1, x_2, \dots)$ and $\Lambda$ is called the weight sequence. For convenience, we use $\lambda B_p$ to denote the constant-weighted backward shift operator on $l^p$, i.e., the associated weight sequence consists of a nonzero constant $\lambda \in \mathbb{C}$. An excellent introduction to the theory of such operators and extensive bibliography can be found in the comprehensive article by Shields [11]. In classical operator theory, we have

**Proposition 1.1.** *Suppose $1 \leq p < \infty$ and $\lambda, \omega \in \mathbb{C}$. Then $\lambda B_p$ and $\omega B_p$ are similar if and only if $|\lambda| = |\omega|$.*

For each $|\lambda| > 1$, $\lambda B_p$ is chaotic by Rolewicz [8] and Grosse-Erdmann [6]. On the other hand, if $|\lambda| \leq 1$, then the orbit of each point in $l^p$ under $\lambda B_p$ approaches to the single point 0 and hence $\lambda B_p$ is not chaotic for $|\lambda| \leq 1$. Therefore,

**Proposition 1.2.** *Suppose $1 \leq p < \infty$, $|\lambda| \leq 1$ and $|\omega| > 1$. Then $\lambda B_p$ and $\omega B_p$ are not topologically conjugate.*

One can see that $2B_p$ and $4B_p$ have almost the same dynamical properties but are not similar. Are there topologically conjugate? Similarly, are $\frac{1}{2}B_p$ and $\frac{1}{4}B_p$ topologically conjugate, and what about $\frac{1}{2}B_p$ and $B_p$? In this article, we'll answer these questions and at the end we'll research in general case, the weight sequence being not constant sequence, by examples.

## 2. Homeomorphisms on $l^p$

In the present section, we'll consider the homeomorphisms on $l^p$. Since $l^p$ is neither compact nor locally compact, the homeomorphisms on $l^p$ would be of some strange properties, for instance, the image of a bounded set under a homeomorphism may be unbounded [13]. However, we also have the following property.

**Lemma 2.1.** *Let $f : l^p \to l^q$ be continuous, where $1 \leq p, q < \infty$. Then for any $x \in l^p$, there is a neighborhood $U$ of $x$ such that $f(U)$ is bounded.*

*Proof.* It is obvious by the continuity of $f$ and the definition of metric on $l^p$.

□

Now let's construct some homeomorphisms on $l^p$.

For any $p \geq 1$ and any $s > 0$, define a map $h_p^{(s)}$ on $l^p$ as follows,

for any $x = (x_1, x_2, \dots) \in l^p$,
$\pi_n \circ h_p^{(s)}(x) = 0$ if $x_n = 0$ and

$$\pi_n \circ h_p^{(s)}(x) = \frac{x_n}{|x_n|} \cdot \sqrt[p]{(\sum_{k=n}^{\infty} |x_k|^p)^s - (\sum_{k=n+1}^{\infty} |x_k|^p)^s}$$

if $x_n \neq 0$. Then one can obtain the following result.

**Proposition 2.2.** *For any $1 \le p < \infty$ and $s > 0$, the map $h_p^{(s)}$ is a homeomorphism from $l^p$ onto itself. In fact, the inverse of $h_p^{(s)}$ is $h_p^{(\frac{1}{s})}$. Moreover, for any positive number $\lambda$, $h_p^{(s)}(\lambda x) = \lambda^s \cdot h_p^{(s)}(x)$for any $x \in l^p$.*

*Proof.* Let $\{x^{(m)}\}_{m=1}^{\infty}$ be a Cauchy sequence in $l^p$ and let $y^{(m)} = h_p^{(s)}(x^{(m)})$ for each $m \ge 1$. Write $x^{(m)} = (x_1^{(m)}, x_2^{(m)}, \dots)$ and $y^{(m)} = (y_1^{(m)}, y_2^{(m)}, \dots)$, for each $m \ge 1$. By the construction of $h_p^{(s)}$, we have

$$\sum_{n=k}^{\infty} |y_n^{(m)}|^p = (\sum_{n=k}^{\infty} |x_n^{(m)}|^p)^s, \quad for\ any\ m, k \in \mathbb{N}. \tag{2.1}$$

So $h_p^{(s)}$ is a map from $l^p$ to $l^p$ and $\{h_p^{(s)}(x^{(m)})\}_{m=1}^{\infty}$ converges by coordinates, i.e., $\{y_n^{(m)}\}_{m=1}^{\infty}$ is a Cauchy sequence for each $n \in \mathbb{N}$. For any $\epsilon > 0$, there is a positive integer $N_0$ such that $\sum\limits_{n=N_0}^{\infty} |x_n^{(m)}|^p < \epsilon$, for each $m \ge 1$. According to (2.1), we have

$$\sum_{n=N_0}^{\infty} |y_n^{(m)}|^p < \epsilon^s, \quad for\ each\ m \ge 1.$$

Consequently, $h_p^{(s)}$ is continuous. Notice that both $h_p^{(s)}$ and $h_p^{(\frac{1}{s})}$ are continuous, and

$$h_p^{(s)} \circ h_p^{(\frac{1}{s})} = h_p^{(\frac{1}{s})} \circ h_p^{(s)} = id.$$

Therefore, $h_p^{(s)}$ is a homeomorphism from $l^p$ onto itself, whose inverse map is $h_p^{(\frac{1}{s})}$. Moreover, it is obvious that for any $x \in l^p$,

$$h_p^{(s)}(\lambda x) = \lambda^s \cdot h_p^{(s)}(x).$$

□

As is well-known (one can see [4]), for any $1 \le p, q < \infty$ there is a natural homeomorphism $g_{pq}$ from $l^p$ onto $l^q$ defined as follows,
for any $x = (x_1, x_2, \dots) \in l^p$,
$\pi_n \circ g_{pq}(x) = 0$ if $x_n = 0$ and

$$\pi_n \circ g_{pq}(x) = \frac{x_n}{|x_n|} \cdot |x_n|^{\frac{p}{q}}$$

if $x_n \neq 0$. And we have

**Proposition 2.3.** *The inverse of $g_{pq}$ is $g_{qp}$, and for any positive number $\lambda$, $g_{pq}(\lambda x) = \lambda^{\frac{p}{q}} \cdot g_{pq}(x)$ for any $x \in l^p$.*

## 3. Topologically conjugate classes for $\lambda B_p$

In this section we'll give the complete topologically conjugate classification for $\lambda B_p$, which indicates that $2B_p$ and $4B_p$, $\frac{1}{2}B_p$ and $\frac{1}{4}B_p$ are topologically conjugate respectively, but not $\frac{1}{2}B_p$ and $B_p$.

For convenience, we define a function $\chi : \mathbb{R} \to \mathbb{R}$ as follows

$$\chi(t) = \begin{cases} 1, & \text{if } t > 1 \\ 0, & \text{if } t = 1 \\ -1, & \text{if } t < 1 \end{cases}$$

**Theorem 3.1.** *Suppose $1 \le p < \infty$ and $\lambda, \omega \in \mathbb{C}$. Then $\lambda B_p$ and $\omega B_p$ are topologically conjugate if and only if $\chi(|\lambda|) = \chi(|\omega|)$.*

*Proof.* Because of Proposition 1.1, it suffices to consider the case that both $\lambda$ and $\omega$ are positive number. If $\chi(|\lambda|) = \chi(|\omega|)$, then there is a positive number $s$ such that $|\lambda|^s = |\omega|$. Consider the homeomorphism $h_p^{(s)}$ defined in the previous section. For any $x = (x_1, x_2, \ldots) \in l^p$, denote $x' = (x_2, x_3, \ldots)$. Then, by the construction of $h_p^{(s)}$ and Proposition 2.2, we have

$$(h_p^{(s)} \circ \lambda B_p)(x) = h_p^{(s)}(\lambda x') = \lambda^s \cdot h_p^{(s)}(x') = \omega \cdot h_p^{(s)}(x'),$$

$$(\omega B_p \circ h_p^{(s)})(x) = \omega \cdot h_p^{(s)}(x').$$

Therefore $h_p^{(s)} \circ \lambda B_p = \omega B_p \circ h_p^{(s)}$ and hence $\lambda B_p$ and $\omega B_p$ are topologically conjugate.

On the converse, by Proposition 1.1 and 1.2 it suffices to verify that $\lambda B_p$ and $B_p$ are not topologically conjugate if $\lambda$ is a positive number less than 1. Now suppose a homeomorphism $f$ conjugates $B_p$ to $\lambda B_p$, where $0 < \lambda < 1$. It follows from Lemma 2.1 that there exist $\delta > 0$ and $M > 0$ such that $\| f(y) \|_p < M$ whenever $\| y \|_p \le \delta$. Write

$$y^{(n)} = (0, 0, \ldots, 0, \underbrace{\delta}_{n^{th}}, 0, \ldots).$$

Since $\| y^{(n)} \|_p = \delta$ for each $n \in \mathbb{N}$, we have $\| (\lambda B_p)^{n-1}(f(y^{(n)}) \|_p \le \lambda^{n-1} M \to 0$ as $n \to \infty$. Consequently,

$$f(y^{(1)}) = \lim_{n\to\infty} f(B_p^{n-1}(y^{(n)})) = \lim_{n\to\infty} (\lambda B_p)^{n-1}(f(y^{(n)})) = 0.$$

Similarly, one can see $f(\frac{y^{(1)}}{2}) = 0$, which is a contradiction.

□

Now we've answer the question in section 1 and it is some surprising that $B_p$ and $\frac{1}{2}B_p$ are not topologically conjugate, although the orbit of each point in $l^p$ under either of them approaches to the single point 0. Furthermore, we can obtain a more general result.

**Theorem 3.2.** *Suppose $1 \le p, q < \infty$ and $\lambda, \omega \in \mathbb{C}$. Then $\lambda B_p$ and $\omega B_q$ are topologically conjugate if and only if $\chi(|\lambda|) = \chi(|\omega|)$.*

*Proof.* According to Theorem 3.1 and Proposition 1.1, it suffices to prove that $\lambda B_p$ and $\omega B_q$ are topologically conjugate if $\chi(\lambda) = \chi(\omega)$ where $\lambda$ and $\omega$ are positive numbers. Now suppose $\lambda$ and $\omega$ are positive numbers with $\chi(\lambda) = \chi(\omega)$. Then $\chi(\omega^{\frac{q}{p}}) = \chi(\lambda) = \chi(\omega)$. Consider the homeomorphism $g_{pq}$ defined in the previous

section. For any $x = (x_1, x_2, \ldots) \in l^p$, denote $x' = (x_2, x_3, \ldots)$. By the construction of $g_{pq}$ and Proposition 2.3, we have

$$g_{pq}(\omega^{\frac{q}{p}} B_p(x)) = g_{pq}(\omega^{\frac{q}{p}} x') = \omega \cdot g_{pq}(x'),$$
$$\omega B_q(g_{pq}(x)) = \omega \cdot g_{pq}(x').$$

Therefore $\omega^{\frac{q}{p}} B_p$ and $\omega B_q$ are topologically conjugate. In addition, $\omega^{\frac{q}{p}} B_p$ and $\lambda B_p$ are topologically conjugate by Theorem 3.1, hence $\lambda B_p$ and $\omega B_q$ are topologically conjugate since topological conjugacy is an equivalent relation.

□

We've got that there are three topological conjugate classes for $\lambda B_p$ in all. At the end of this paper, we'll consider the general case that the weight sequence is not constant sequence, and we would see some new topologically conjugate classes different from the three classes.

First of all, we give or restate in following Proposition 3.3 the characterizations of several topologically conjugate invariances such as chaoticity, transitivity and strong mixing for weighted backward shift operators. Denote $\beta(n)$ as

$$\beta(n) = \prod_{i=1}^{n} \omega(i), \quad for \quad n = 1, 2, \ldots \quad ,$$

where $\{\omega_n\}_{n=1}^{\infty}$ is a weight sequence.

**Proposition 3.3.** *If $T$ is a weighted backward shift operator on $l^p$, $1 \leq p < \infty$, with weight sequence $\{\omega_n\}_{n=1}^{\infty}$, then*

*(I)(K. G. Grosse-Erdmann [6]) $T$ is chaotic if and only if $\sum\limits_{n=1}^{\infty} \frac{1}{|\beta(n)|^p} < \infty$;*

*(II)(G. Costakis and M. Sambarino [7]) $T$ is strongly mixing if and only if $\lim\limits_{n\to\infty} |\beta(n)| = \infty$.*

*(III)(H. N. Salas [9]) $T$ is transitive if and only if $\limsup\limits_{n\to\infty} |\beta(n)| = \infty$.*

**Example 3.4.** Let $T^{(1)}$ be a weighted backward shift operator on $l^2$ with the weight sequence $\{\omega_n^{(1)}\}_{n=1}^{\infty}$, where

$$\{\omega_1^{(1)}\} = 1 \ \ and \ \ \{\omega_n^{(1)}\} = \sqrt{\frac{n}{n-1}} \ \ for \ \ n \geq 2.$$

It implies $\beta(n) = \sqrt{n}$, for $n \geq 1$. Then we have $\sum\limits_{n=1}^{\infty} \frac{1}{|\beta(n)|^2} = \infty$ and $|\beta(n)| \to \infty$ as $n \to \infty$. Consequently, the operator $T^{(1)}$ is strongly mixing but not chaotic. Therefore $T^{(1)}$ is not topologically conjugate to $\lambda B_2$ for every $\lambda \in \mathbb{C}$.

**Example 3.5.** Let $T^{(2)}$ be a weighted backward shift operator on $l^2$ with the weight sequence $\{\omega_n^{(2)}\}_{n=1}^{\infty}$, where

$$(\omega_1^{(2)}, \omega_2^{(2)}, \ldots) = (2, \frac{1}{2}, 2, 2, \frac{1}{2}, \frac{1}{2}, \underbrace{2, 2, 2}_{3}, \underbrace{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}_{3}, \ldots, \underbrace{2, 2, \ldots, 2}_{k}, \underbrace{\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}}_{k}, \ldots).$$

It is easy to see that $T^{(2)}$ is transitive but not strongly mixing and hence $T^{(2)}$ is not topologically conjugate to either $\lambda B_2$ or $T^{(1)}$.

**Example 3.6.** Let $T^{(3)}$ be a weighted backward shift operator on $l^2$ with the weight sequence $\{\omega_n^{(3)}\}_{n=1}^{\infty}$, where

$$(\omega_1^{(3)}, \omega_2^{(3)}, \dots) = (\frac{1}{2}, 2, \frac{1}{2}, \frac{1}{2}, 2, 2, \underbrace{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}_{3}, \underbrace{2, 2, 2}_{3}, \dots, \underbrace{\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2}}_{k}, \underbrace{2, 2, \dots, 2}_{k}, \dots).$$

It is easy to see that $T^{(3)}$ is not transitive. Now consider the point

$$x = (0, 1, 0, 0, 0, \frac{1}{2}, \underbrace{0, 0, 0}_{3}, \underbrace{0, 0, \frac{1}{2^2}}_{3}, \dots, \underbrace{0, 0, \dots, 0, 0}_{k}, \underbrace{0, 0, \dots, 0, \frac{1}{2^{k-1}}}_{k}, \dots).$$

We have $\| (T^{(3)})^n(x) \|_2 \geq 1$ for each $n \geq 1$. Consequently, $T^{(3)}$ is not topologically conjugate to $\lambda B_2$ for $|\lambda| \leq 1$. Therefore, $T^{(3)}$ is not topologically conjugate to $\lambda B_2$ or $T^{(1)}$, or $T^{(2)}$.

## References


[1] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey, On Devaney's definition of chaos, *Amer. Math. Monthly, Vol. 99* 1992, 332-334.

[2] L. S. Block and W. A. Coppel, Dynamics in one dimension, *Lecture Notes in Mathematics, 1513* Springer, 1992.

[3] R. L. Devaney, An introduction to chaotic dynamical systems, Addison Weseley, 1987.

[4] J. J. Dijkstra and J. van Mill, Topological equivalence of discontinuous norms, *Israel J. Math. Vol. 128* 2002, 177-196.

[5] K. G. Grosse-Erdmann, Recent developments in hypercyclicity , *Rev. R. Acad. Cien. Serie A. Mat., RACSAM Vol. 97 (2)* 2003, 273-286.

[6] K. G. Grosse-Erdmann, Hypercyclic and chaotic weighted shifts , *Studia Math., Vol. 139 (1)* 2000, 47-68.

[7] G. Costakis and M. Sambarino, Topologically mixing hypercyclic operators, *Proc. Amer. Math. Soc. Vol. 132 (2)* 2003, 385-389.

[8] S. Rolewicz, On orbits of elements, *Studia Math., Vol. 32* 1969, 17-22.

[9] H. N. Salas, Hypercyclic weighted shifts, *Trans. Amer. Math. Soc. Vol. 347 (3)* 1995, 993-1004.

[10] J. H. Shapiro, Notes on Dynamics of Linear Operators, unpublished Lecture Notes, available at http://www.math.msu.edu/˜shapiro.

[11] A. L. Shields, Weighted shift operators and analyic function theory, *"Topics in Operator Theory", Mathematical Surveys, Vol. 13* American Mathematical Society, Providence, RI.

[12] S. Silverman, On maps with dense orbits and the definition of chaos, *Rocky Mountain Jour. Math., Vol. 22* 1992, 353-375.

[13] C. C. Tseng and N. C. Wong , Invertibility in infinite-dimensional spaces, *Proc. Amer. Math. Soc. Vol. 128 (2)* 1999, 573-581.



Bingzhe Hou, Institute of Mathematics , Jilin University, 130012, Changchun, P.R.China

*E-mail address*: `abellengend@163.com`

Gongfu liao, Department of Mathematics , Jilin Normal University, 136000, Siping, P.R.China

*Current address*: Institute of Mathematics , Jilin University, 130012, Changchun, P.R.China

*E-mail address*: `liaogh@email.jlu.edu.cn`

Yang Cao, Institute of Mathematics , Jilin University, 130012, Changchun, P.R.China

*E-mail address*: `caoyang@jlu.edu.cn`